

1.

$$\iint_S xdy \wedge dz + ydz \wedge dx + zdx \wedge dy,$$

S – верхняя сторона поверхности $\frac{x}{2} + \frac{y}{3} + \frac{z}{4} = 1, x, y, z \geq 0$

$$\begin{aligned} \iint_S zdx \wedge dy &= \iint_{x,y \geq 0, \frac{x}{2} + \frac{y}{3} \leq 1} 4 \left(1 - \frac{x}{2} - \frac{y}{3} \right) dx dy = 6 \iint_{u,v \geq 0, u+v \leq 1} 4(1-u-v) dudv = \\ &= 24 \int_0^1 du \int_0^{1-u} (1-u-v) dv = 24 \int_0^1 \frac{(1-u)^2}{2} du = 24 \cdot \frac{1}{2} \cdot \frac{1}{3} = 4 \end{aligned}$$

Ответ: 12.

Или

$$\begin{aligned} \iint_S xdy \wedge dz + ydz \wedge dx + zdx \wedge dy &= \iint_S (x \cos \lambda + y \cos \mu + z \cos \nu) d\sigma = \\ &= \iint_S \frac{6x+4y+3z}{\sqrt{36+16+9}} d\sigma = \frac{12}{\sqrt{61}} \iint_S d\sigma = \frac{12}{\sqrt{61}} \iint_{x,y \geq 0, \frac{x}{2} + \frac{y}{3} \leq 1} \sqrt{1+4+\frac{16}{9}} dx dy = \\ &= \frac{12}{\sqrt{61}} \frac{\sqrt{61}}{3} \iint_{x,y \geq 0, \frac{x}{2} + \frac{y}{3} \leq 1} dx dy = 4 \cdot 3 = 12 \end{aligned}$$

$$\frac{x}{2} + \frac{y}{3} + \frac{z}{4} = 1, 6x + 4y + 3z = 12$$

2.

$$\iint_S x^3 dy \wedge dz + y^3 dz \wedge dx + z^3 dx \wedge dy,$$

S – внешняя сторона сферы $x^2 + y^2 + z^2 = a^2$

Решение_1.

$$\begin{aligned} \iint_S z^3 dx \wedge dy &= \iint_{S, z \geq 0} z^3 dx \wedge dy + \iint_{S, z \leq 0} z^3 dx \wedge dy = 2 \iint_{x^2+y^2 \leq a^2} (a^2 - x^2 - y^2)^{3/2} dx dy = \\ &= 4\pi \int_0^a (a^2 - r^2)^{3/2} r dr = -4\pi \cdot \frac{1}{5} (a^2 - r^2)^{5/2} \Big|_0^a = \frac{4}{5} \pi a^5 \end{aligned}$$

Решение_2.

$$\begin{aligned} \iint_S x^3 dy \wedge dz + y^3 dz \wedge dx + z^3 dx \wedge dy &= \iint_S \left(x^3 \frac{x}{a} + y^3 \frac{y}{a} + z^3 \frac{z}{a} \right) d\sigma = \\ &= \iint_S (x^4 + y^4 + z^4) d\sigma = 6 \iint_S z^4 d\sigma = 6 \iint_{x^2+y^2 \leq a^2} (a^2 - x^2 - y^2)^2 \frac{dxdy}{\sqrt{a^2 - x^2 - y^2}} = \\ &= 6 \iint_{x^2+y^2 \leq a^2} (a^2 - x^2 - y^2)^{3/2} dxdy = 12\pi \int_0^a (a^2 - r^2)^{3/2} r dr = \frac{12}{5} \pi a^5. \end{aligned}$$

Решение_3.

$$\begin{aligned} \iint_S x^3 dy \wedge dz + y^3 dz \wedge dx + z^3 dx \wedge dy &= 3 \iiint_{x^2+y^2+z^2 \leq a^2} (x^2 + y^2 + z^2) dxdydz = \\ &= 3 \cdot 2\pi \int_{-\pi/2}^{\pi/2} \cos \psi d\psi \int_0^a \rho^5 d\rho = 3 \cdot 2\pi \cdot 2 \cdot \frac{1}{5} a^5 = \frac{12}{5} \pi a^5. \end{aligned}$$

Ответ: $\frac{12}{5} \pi a^5$.

3.

$$\iint_S z^2 dx \wedge dy, S - \text{внешняя сторона поверхности конуса } x^2 + y^2 \leq z^2, 0 \leq z \leq 1.$$

Решение_1.

$$\begin{aligned} \iint_S z^2 dx \wedge dy &= \iint_{S, z=1} z^2 dx \wedge dy + \iint_{S, x^2+y^2=1} z^2 dx \wedge dy; \\ \iint_{S, z=1} z^2 dx \wedge dy &= \iint_{x^2+y^2 \leq 1} dxdy = \pi, \quad \iint_{S, x^2+y^2=1} z^2 dx \wedge dy = - \iint_{x^2+y^2 \leq 1} (x^2 + y^2) dxdy = -\frac{\pi}{2}. \end{aligned}$$

Ответ: $\frac{\pi}{2}$.

Решение_2.

$$\begin{aligned} \iint_S z^2 dx \wedge dy &= 2 \iiint_{\substack{0 \leq z \leq 1 \\ x^2+y^2 \leq z^2}} z dxdydz = 2 \int_0^1 z dz \iint_{x^2+y^2 \leq z^2} dxdy = \\ &= 2\pi \int_0^1 z^3 dz = \frac{\pi}{2} \end{aligned}$$

4.

$$\iint_S (x^3 + z^3) dy \wedge dz, S - \text{внешняя сторона поверхности конуса } x^2 + y^2 \leq z^2, 0 \leq z \leq 1.$$

$$\begin{aligned}
\iint_S (x^3 + z^3) dy \wedge dz &= \iint_{S, z=1} (x^3 + z^3) dy \wedge dz + \iint_{S, x^2+y^2=z^2} (x^3 + z^3) dy \wedge dz = \\
&= \iint_{S, x^2+y^2=z^2} (x^3 + z^3) dy \wedge dz = \iint_{S, x^2+y^2=z^2, x \geq 0} (x^3 + z^3) dy \wedge dz + \iint_{S, x^2+y^2=z^2, x \leq 0} (x^3 + z^3) dy \wedge dz = \\
&= \iint_{\substack{0 \leq z \leq 1 \\ -z \leq y \leq z}} \left((z^2 - y^2)^{3/2} + z^3 \right) dy dz - \iint_{\substack{0 \leq z \leq 1 \\ -z \leq y \leq z}} \left(-(z^2 - y^2)^{3/2} + z^3 \right) dy dz = \\
&= 2 \iint_{\substack{0 \leq z \leq 1 \\ -z \leq y \leq z}} (z^2 - y^2)^{3/2} dy dz = 4 \iint_{\substack{0 \leq z \leq 1 \\ 0 \leq y \leq z}} (z^2 - y^2)^{3/2} dy dz = \\
&= 4 \int_0^1 dz \int_0^z (z^2 - y^2)^{3/2} dy = 4 \int_0^1 z^4 dz \int_0^{\pi/2} \cos^4 t dt = \frac{4}{5} \cdot \frac{3!!}{4!!} \cdot \frac{\pi}{2} = \frac{3}{20} \pi.
\end{aligned}$$

Ответ: $\frac{3}{20} \pi$.

Решение_2.

$$\begin{aligned}
\iint_S (x^3 + z^3) dy \wedge dz &= 3 \iiint_{\substack{0 \leq z \leq 1 \\ x^2+y^2 \leq z^2}} x^2 dx dy dz = 3 \int_0^1 dz \iint_{x^2+y^2 \leq z^2} x^2 dx dy = 3 \int_0^1 dz \int_0^{2\pi} \cos^2 \varphi d\varphi \int_0^z r^3 dr = \\
3\pi \int_0^1 \frac{z^4}{4} dz &= \frac{3}{20} \pi
\end{aligned}$$

5.

$$\iint_S (x^4 + z^4) dx \wedge dy,$$

S – внешняя сторона поверхности цилиндра $x^2 + y^2 = a^2$. $0 \leq z \leq b$.

Решение_1.

$$\begin{aligned}
\iint_S (x^4 + z^4) dx \wedge dy &= \iint_{S, z=0} (x^4 + z^4) dx \wedge dy + \iint_{S, z=b} (x^4 + z^4) dx \wedge dy + \iint_{S, x^2+y^2=a^2} (x^4 + z^4) dx \wedge dy = \\
&= - \iint_{x^2+y^2 \leq a^2} x^4 dx dy + \iint_{x^2+y^2 \leq a^2} (x^4 + b^4) dx dy = \iint_{x^2+y^2 \leq a^2} b^4 dx dy = \pi a^2 b^4
\end{aligned}$$

Решение_2.

$$\iint_S (x^4 + z^4) dx \wedge dy = 4 \iiint_{\substack{0 \leq z \leq b \\ x^2+y^2 \leq z^2}} z^3 dx dy dz = 4 \int_0^b z^3 dz \iint_{x^2+y^2 \leq z^2} dx dy = \pi a^2 b^4$$

Ответ: $\pi a^2 b^4$

6.

$$\iint_S (x^4 + y^4 + y^5) dz \wedge dx,$$

S – внешняя сторона поверхности цилиндра $x^2 + y^2 = a^2$. $0 \leq z \leq b$

Решение_1.

$$\begin{aligned}
 \iint_S (x^4 + y^4 + y^5) dz \wedge dx &= \iint_{S, x^2+y^2=a^2} (x^4 + y^4 + y^5) dz \wedge dx = \\
 &= \iint_{\substack{0 \leq z \leq b \\ -a \leq x \leq a}} \left(x^4 + (a^2 - x^2)^2 + (a^2 - x^2)^{5/2} \right) dz dx - \iint_{\substack{0 \leq z \leq b \\ -a \leq x \leq a}} \left(x^4 + (a^2 - x^2)^2 - (a^2 - x^2)^{5/2} \right) dz dx \\
 &= 2 \iint_{\substack{0 \leq z \leq b \\ -a \leq x \leq a}} (a^2 - x^2)^{5/2} dz dx = 4 \iint_{\substack{0 \leq z \leq b \\ 0 \leq x \leq a}} (a^2 - x^2)^{5/2} dz dx = 4 \int_0^b dz \int_0^a (a^2 - x^2)^{5/2} dx = \\
 &= 4a^6 b \int_0^{\pi/2} \cos^6 t dt = 4a^6 b \cdot \frac{5!!}{6!!} \cdot \frac{\pi}{2} = \frac{5}{8} \pi a^6 b.
 \end{aligned}$$

Решение_2.

$$\begin{aligned}
 \iint_S (x^4 + y^4 + y^5) dz \wedge dx &= \iiint_{\substack{0 \leq z \leq b \\ x^2+y^2 \leq a^2}} (4y^3 + 5y^4) dx dy dz = \\
 &= 5 \int_0^b dz \iint_{x^2+y^2 \leq a^2} y^4 dx dy = 5b \int_0^{2\pi} \sin^4 \varphi d\varphi \int_0^a r^5 dr = 5b \cdot 4 \cdot \frac{3\pi}{16} \cdot \frac{a^6}{6} = \frac{5}{8} \pi a^6 b
 \end{aligned}$$

7.

$$\iint_S (x^5 + z^5) dy \wedge dz, S - \text{внешняя сторона поверхности конуса } x^2 + y^2 \leq 4z^2, 0 \leq z \leq 1.$$

Решение_1.

$$\begin{aligned}
 \iint_S (x^5 + z^5) dy \wedge dz &= \iint_{S, x^2+y^2=4z^2} (x^5 + z^5) dy \wedge dz = \\
 &= \iint_{\substack{0 \leq z \leq 1 \\ x^2+y^2=4z^2}} \left((4z^2 - y^2)^{5/2} + z^5 \right) dy dz - \iint_{\substack{0 \leq z \leq 1 \\ x^2+y^2=4z^2}} \left((4z^2 - y^2)^{5/2} + z^5 \right) dy dz = \\
 &= 2 \iint_{\substack{0 \leq z \leq 1 \\ -2z \leq y \leq 2z}} (4z^2 - y^2)^{5/2} dy dz = 4 \int_0^1 dz \int_0^{2z} (4z^2 - y^2)^{5/2} dy = 4 \int_0^1 64z^6 dz \int_0^{2\pi} \cos^6 t dt = \\
 &= 4 \cdot 64 \cdot \frac{1}{7} \cdot \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6} \cdot \frac{\pi}{2} = \frac{40}{7} \pi.
 \end{aligned}$$

Решение_2.

$$\begin{aligned}
 \iint_S (x^5 + z^5) dy \wedge dz &= 5 \iiint_{\substack{0 \leq z \leq 1 \\ x^2+y^2 \leq 4z^2}} x^4 dx dy dz = 5 \int_0^1 dz \iint_{x^2+y^2 \leq 4z^2} x^4 dx dy = \\
 &= 5 \int_0^1 dz \int_0^{2\pi} \cos^4 \varphi d\varphi \int_0^{2z} r^5 dr = 5 \int_0^1 \frac{64}{6} z^6 dz \frac{3}{4} \pi = \frac{40}{7} \pi
 \end{aligned}$$

$$8. \iint_S (x^4 + z^4) dx \wedge dy, S - \text{внешняя сторона поверхности конуса } x^2 + y^2 \leq z^2, 0 \leq z \leq 3.$$

Решение_1.

$$\begin{aligned}
\iint_S (x^4 + z^4) dx \wedge dy &= \iint_{S, z=3} (x^4 + z^4) dx \wedge dy - \iint_{S, x^2+y^2=z^2} (x^4 + z^4) dx \wedge dy = \\
&= \iint_{x^2+y^2 \leq 9} (x^4 + 81) dx dy - \iint_{x^2+y^2 \leq 9} (x^4 + (x^2 + y^2)^2) dx dy = \\
&= \iint_{x^2+y^2 \leq 9} (81 - (x^2 + y^2)^2) dx dy = 729\pi - \iint_{x^2+y^2 \leq 9} (x^2 + y^2)^2 dx dy = \\
&= 729\pi - 2\pi \int_0^3 r^5 dr = 729\pi - 2\pi \cdot \frac{729}{6} = 486\pi.
\end{aligned}$$

Решение_2.

$$\iint_S (x^4 + z^4) dx \wedge dy = 4 \iiint_{\substack{0 \leq z \leq 3 \\ x^2+y^2 \leq z^2}} z^3 dx dy dz = 4 \int_0^3 z^3 dz \iint_{x^2+y^2 \leq z^2} dx dy = 4\pi \int_0^3 z^5 dz = 4\pi \frac{729}{6} = 486\pi.$$